Types of Balanced Growth

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Abstract. The notions of semi-endogenous and non-scale growth are sometimes used interchangeably. This paper shows that they are logically independent. It is proven that the existence of a steady state generally depends on knife-edge conditions. Choosing a linear population equation implies that only models of semi-endogenous growth without scale-effects do not depend on further knife-edge conditions.

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1 Introduction

Romer's (1990) model of endogenous technological change has been criticized by Jones (1995) for the involved scale-effects of the size of the economy (measured by population size, e.g.). Jones' model does not involve these scale-effects and he calls it a model of *semi-endogenous* growth, because technological change is endogenous while the steady state growth rates are pinned down by the exogenous rate of population growth, largely independent of policy measures. Subsequently, models of this kind have also been denoted as *non-scale growth models* (e.g. Eicher and Turnovsky, 1999; Jones, 1999). While it is obvious from considering the so-called second generation of non-scale models (e.g. Peretto, 1998) or even the pioneering Uzawa-Lucas model (Lucas, 1988, sec. 4) that a non-scale model need not be a model of semi-endogenous growth, a clear-cut distinction of these notions is missing.

Abstracting from the microeconomic foundations of R&D and other sources of growth, this paper introduces a simple but general model in order to prove the following assertions. First, any model of balanced growth requires a knife-edge condition to be met. A justifiable knife-edge condition is the assumption that the growth rate of population is constant. Second, only semi-endogenous non-scale growth models do not depend on further knife-edge restrictions. Third, the absence of scale-effects is neither necessary nor sufficient for growth to be semi-endogenous.

Apart from the third assertion, these results are known, but only proven for special cases or hidden behind sophisticated models (cf. e.g. Eicher and Turnovsky, 1999;

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Jones, 1999, 2001; Li, 2000). It is the purpose of the present paper to give a simple, unified treatment of these issues in a general setting. E.g., while the knife-edge condition is usually formulated with respect to a particular parameter of a parameterized model, Proposition 1 below provides a general criterion in terms of a determinant.

2 The Model

With respect to the following definitions it is assumed that economic policy cannot influence the growth rate of population and the parameters of the production functions. As usual, however, it can affect resource allocation (including savings) by taxes and subsidies, e.g. A *balanced growth path* (or *steady state*) is a path where all variables grow at constant rates. Growth involves (no) *scale-effects* if long-run per capita growth rates (do not) vary with the size of the economy as measured by its population. Growth is *endogenous* if long-run per capita growth rates are positive without exogenous technical progress and sensitive to economic policy affecting resource allocation. Growth is *semi-endogenous* if long-run per capita growth rates are positive without exogenous technical change but insensitive to economic policy.

Let Y be the output of the final good, A the stock of technology, L the population (labor force), and K the stock of physical capital. The final good can be used for consumption as well as for capital accumulation. Consider the following general three-sector production structure:

$$Y = F(k_Y K, a_Y A, l_Y L), \tag{1}$$

$$\dot{A} = G(k_A K, a_A A, l_A L), \tag{2}$$

$$\dot{L} = H(k_L K, a_L A, l_L L), \tag{3}$$

where $\sum_i x_i = 1$ ($x = k, a, l$; $i = Y, A, L$) if the respective input is private (rival in use). In this case, x_i is the fraction of the private input devoted to the production of output i. However, inputs may as well be public goods. E.g., if $a_i = 1\ \forall i$, A is a pure public input, which should be interpreted as general knowledge, while it could be human capital as a private input. In-between cases are also possible. E.g., if $l_Y + l_A = 1$ and $l_L = 1$, labor would as a private input be allocated to the production of Y and $\dot{A}$, while it could at the same time be used as an input for the *production* of children. Equations (1)–(3) generalize the production structure analyzed by Eicher and Turnovsky (1999) by allowing for the possibility of external effects of all three factors of production and adding a kind of *production function* for labor.

As the growth rate $g_L := \dot{L}/L$ of population equals the difference between the birth rate b and the mortality rate d, the function H must satisfy the condition

$$H(k_L K, a_L A, l_L L) \equiv (b - d)L. \tag{4}$$

Both b and d may depend on the stock of technology A and on the resources $l_L L$ and $k_L K$ devoted to child bearing and medical care, e.g. In case of an optimizing solution with respect to fertility, the optimum birth rate in feedback form will also depend on the state variables of the model.[1] The particular reason for the dependency of H on K, A and L is irrelevant with respect to the following analysis, however.

[1] Endogenous fertility is analyzed by Becker and Barro (1988) and Jones (2001), e.g.

Turning to the demand side, it suffices to assume that the consumption function implies a constant steady state consumption rate, $c = C/Y$ (this will be the case for optimum consumption paths as well as for constant savings rates, e.g.). Neglecting the depreciation of capital, the capital accumulation equation $\dot{K} = Y - C$ and the constancy of

$$g_K := \frac{\dot{K}}{K} = \frac{Y}{K} - \underbrace{\frac{C}{Y}}_{=c<1} \frac{Y}{K}$$

then imply that $g_Y = g_K = g_C$ (all growth rates are denoted by g). Since all variables grow at constant rates in a steady state, all x_i fractions in (1)–(3) must be constant. Using $g_Y = g_K$, $g_A = g_{\dot{A}}$, and $g_L = g_{\dot{L}}$ in case of constant growth rates, logarithmic differentiation of equations (1)–(3) along a balanced growth path yields

$$\begin{pmatrix} (1-\sigma_K) & -\sigma_A & -\sigma_L \\ -\eta_K & (1-\eta_A) & -\eta_L \\ -\mu_K & -\mu_A & (1-\mu_L) \end{pmatrix} \begin{pmatrix} g_K \\ g_A \\ g_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5}$$

where σ_i, η_i, and μ_i, $i = K, A, L$, are the non-negative output elasticities of capital, knowledge, and labor with respect to the functions F, G and H, respectively.

It is a standard result of linear algebra that a homogeneous system of linear equations such as (5) has only the trivial solution $g_K = g_A = g_L = 0$ if $|J| \neq 0$, where J is the matrix on the left hand side of equation (5). Hence, a positive solution with respect to the growth rates is impossible unless $|J| = 0$. But $|J| = 0$ is just a general knife-edge condition pertaining to the output elasticities.[2] This result yields

Proposition 1 *A steady state with positive growth rates does not exist unless the knife-edge condition $|J| = 0$ is met.*

In view of Proposition 1, one needs a good justification of the consideration of balanced growth models. Most existing models assume that a particular equation of (1)–(3) is linear in order to get $|J| = 0$. As Jones (2001) has pointed out, with the exception of those assumptions pertaining to the population equation, these assumptions are completely ad hoc. E.g., Lucas (1988) uses a linear equation for $\dot{A}$, implying that doubling the stock of human capital, A, leads to doubling the change in human capital. "If a 7th grader and a high school graduate each go to school for 8 hours per day, does the high school graduate learn twice as much?" (Jones, 2001, p. 12). Similar objections pertain to the linearity assumptions in other models, e.g. those following Romer (1990), where A denotes the stock of existing ideas or designs.

In contrast, if individuals choose a particular number of children, the total number of children clearly doubles if population doubles. Thus, linearity in the population equation results from the standard replication argument. Although the function $b - d = H(k_L K, a_L A, l_L L)/L$ is not constant in the general formulation in (3), it must be noted that the number of children per family is bounded from above and below by nature.

[2] A rigorous argument proceeds as follows. The set of all square matrices in R^n with non-zero determinant is a dense open subset of the set of all square matrices in R^n (cf. e.g. Hirsch and Smale, 1974, p. 157). Thus, a square matrix with zero determinant can always be turned into a matrix with non-zero determinant by an arbitrarily slight perturbation of its entries.

Also, history teaches us that the birth rate may well exceed the death rate for long time horizons and that exponential population growth is possible. Defining $n := b - d$, the assumption that n is constant therefore seems to be a justifiable abstraction from reality.[3] Equation (3) then reads $\dot{L} = nL$, which is a knife-edge condition requiring linearity in L.

In this case, equations (1)–(3) may be simplified by dropping (3) and using $g_L = n$ in (1) and (2). Logarithmic differentiation then yields

$$\begin{pmatrix} (1-\sigma_K) & -\sigma_A \\ -\eta_K & (1-\eta_A) \end{pmatrix} \begin{pmatrix} g_K \\ g_A \end{pmatrix} = \begin{pmatrix} \sigma_L n \\ \eta_L n \end{pmatrix}, \tag{6}$$

the system analyzed by Eicher and Turnovsky (1999). If the right hand side has at least one positive element (i.e., $n > 0$ and $\sigma_L > 0$ and/or $\eta_L > 0$), this system of equations in g_K and g_A has a unique solution if and only if $|I| \neq 0$, where I is the matrix on the left hand side of (6). This solution is

$$g_K = \underbrace{\frac{\sigma_L(1-\eta_A)+\eta_L\sigma_A}{|I|}}_{=:\gamma_K} n = \gamma_K n, \quad g_A = \underbrace{\frac{\eta_L(1-\sigma_K)+\sigma_L\eta_K}{|I|}}_{=:\gamma_A} n = \gamma_A n. \tag{7}$$

Using the non-negativity of output elasticities, Eicher and Turnovsky (1999) show that if the right hand side of (6) is positive, the steady state growth rates of Y, C, K, and A are positive if and only if $|I| > 0$ and $\sigma_K < 1$. They also provide sufficient conditions for $\gamma_K > 1$, which implies that $g_y = g_Y - g_L = (\gamma_K - 1)n > 0$, where $y = Y/L$.

Regarding the present discussion, it is important to observe that the growth rates in (7) are determined by the exogenous growth rate n of population and output elasticities alone, which follows solely from $|I| \neq 0$. Thus, if $\gamma_K > 1$, growth is semi-endogenous and involves no scale effects. Only if the additional knife-edge condition $|I| = 0$ was met, growth could be endogenous and/or involve scale-effects. This proves

Proposition 2 *If the growth rate of population is exogenous, endogenous growth and/or growth with scale-effects is impossible unless the additional knife-edge condition* $|I| = 0$ *is met.*

As it is straightforward to generalize criterions in terms of determinants to higher dimensions, the arguments leading to Propositions 1 and 2 would be equally valid in case of two knowledge indices, A_1 and A_2, say. The analysis therefore also applies to the second generation of non-scale models with two R&D sectors.

All conditions refer to the steady state, which presupposes that γ_K and γ_A in (7) are constant. Sufficient conditions are: First, constant returns to scale with respect to both production functions, second, both production functions are Cobb-Douglas, and third, a separability condition (cf. Eicher and Turnovsky, 1999, p. 404), which, however, does not add too much generality compared to the Cobb-Douglas case.

[3] It is understood that there are plausible cases in which the assumption that n is constant is not met. E.g., if the birth rate asymptoted to the death rate, n would be decreasing in time and growth would be unbalanced by definition. However, as $n \to 0$ for $t \to \infty$, there could be an asymptotic steady state in which all growth rates are zero, cf. equations (7) below. As long as $t < \infty$, the dynamics off the steady state would have to be considered, however. Considering balanced growth therefore requires a simplifying assumption such as a constant n.

3 Special Cases

As shown by Eicher and Turnovsky (1999), the model described by (6) comprises the production sectors of several well-known models of economic growth as special cases. Among these are the models of Romer (1990), Jones (1995), and the Uzawa-Lucas model (Lucas, 1988, Sec. 4). While the Romer model is endogenous with scale effects, the Jones model is of the non-scale semi-endogenous type. The Uzawa-Lucas model generates endogenous growth without scale effects. The only type of model that is not known by now is a model of semi-endogenous growth with scale effects. Thus, the logical independence of the notions of semi-endogenous and non-scale growth is proven by giving an example of such a model.

The following is not to be understood as a realistic description of actual growth processes, but just as an example showing the logical possibility of semi-endogenous growth with scale effects. Let

$$Y = K^{\alpha_1}(AL)^{\alpha_2}, \quad 0 < \alpha_1, \alpha_2 < 1$$
$$\dot{A} = \psi AL, \quad \psi > 0.$$

Observe that $|I| = 0$. The second equation implies $g_A = \psi L$. Thus, a steady state does not exist unless $n = 0$. In this case, the first equation implies $g_Y = g_K = \alpha_2 g_A/(1-\alpha_1)$, which together with $g_A = \psi L$ and $y = Y/L$ yields

$$g_y = g_Y = \frac{\alpha_2}{1-\alpha_1}\psi L.$$

As there is no possibility of raising this positive growth rate by e.g. a reallocation of labor, growth is semi-endogenous. Nevertheless, apparently there is a scale-effect of the population size. This proves

Proposition 3 *The notions of* semi-endogenous growth *and* non-scale growth *are logically independent.*

4 Conclusion

A balanced growth path does not exist without any knife-edge conditions. The only such condition that appears to be justifiable is $\dot{L} = nL$. Accepting this condition implies that a robust model of economic growth must be of the semi-endogenous non-scale type, although these notions are logically independent. Thus, if we want to analyze models of economic growth with steady states, we must be prepared to accept that the result with respect to per capita growth rates of income is of the form

$$g_y = \text{constant} \cdot n, \tag{8}$$

where the constant is not subject to political influence.

This result involves two principal problems. First, it implies the counterfactual prediction that the growth rate of per capita income always increases in the growth rate of population. This *population puzzle* can be solved by considering open economy models of non-scale growth (cf. Christiaans, 2003). Second, semi-endogenous growth is a sobering implication. However, if trade policy can influence the pattern of international specialization and if growth potentials in different production sectors differ, trade policy should affect long-run growth rates. These issues await a further clarification. Another approach is the consideration of endogenous fertility, turning n into an endogenous variable that may be affected by economic policy (cf. Jones, 2001).

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