Appendix C to: The Implications of Rybczynski's Theorem for Government Spending, Learning by Doing, and Labor Mobility

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Abstract. This appendix sketches the derivation of Fig. 2 in *The Implications of Rybczynski's Theorem for Government Spending, Learning by Doing, and Labor Mobility* with paper and pencil. This procedure shows that a fairly complete picture of the possible time paths of labor and knowledge can be obtained and moreover helps to get a better understanding of the mechanics of the model. Numbers of equations and figures without a leading "C" refer to the original paper. Notation and assumptions coincide and will not be explained again.

1 The Pattern of Specialization

For any given positive value of v the region will specialize in the production of good 2 if $q = 0$. As pq^n enters the problem of revenue maximization (1) just as a relative price, it is straightforward that in equilibrium the marginal rate of transformation between the activity levels, $-dz_2/dz_1$, equals pq^n if production is diversified; if it is greater (smaller) than pq^n, the region specializes in the production of good 2 (good 1). (For the sake of brevity, *specialization* will always mean *complete specialization* in the sequel.) As Herberg (1969) has shown, $-dz_2/dz_1$ is positive and finite for all admissible values $0 \leq z_1 \leq z_1^{\max}$ under the present assumptions ($z_1^{\max}$ is defined by $z_1^{\max} := f_1(v,c)$). Thus, for any finite and positive p, specialization in the production of good 2 occurs at a positive value of q.

Now consider rising values of q. Since p is fixed and $-dz_2/dz_1$ is finite, it is always possible to choose a q to get $-dz_2/dz_1 < pq^n$ and therefore specialization in the production of good 1. Imagining a horizontal line in (q,v)-space, one then starts at $q = 0$ producing only good 2, reaches a point of diversification at a $q > 0$, and eventually reaches the point of specialization in the production of good 1 at an even greater value of q. (Note that $-dz_2/dz_1$ takes on all values between the minimum and the maximum rate of transformation between the activity levels, which are different due to different factor intensities in both sectors.) Hence, for each given value of v, there are corresponding values of q where specialization in the production of good 2 or good 1 occurs, respectively. The geometrical loci, denoted by $S(2)$ and $S(1)$ respectively, that separate specialization from diversification are the implicit functions defined by $x_1 = q^n Z^1(pq^n, v, c) = 0$ and $x_2 = Z^2(pq^n, v, c) = 0$, evaluated at the points of transition from diversification to specialization. Using appropriate one-sided derivatives, the slopes of these lines may be calculated as

$$\left.\frac{dv}{dq}\right|_{S(2)} = -n\frac{pX_p^1}{qX_v^1} > 0,$$

$$\left.\frac{dv}{dq}\right|_{S(1)} = -n\frac{pX_p^2}{qX_v^2} > 0.$$

(It follows from $x_1 = q^n Z^1(pq^n, v, c)$ that $X_q^1 = n(x_1 + pX_p^1)/q$; noting that $x_1 = 0$ on $S(2)$, the first formula is derived. The second formula follows similarly.) The above discussion implies that the curve $S(1)$ lies below the curve $S(2)$.

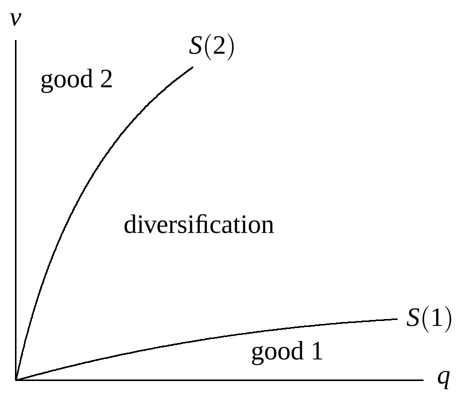

Fig. C1. The pattern of specialization

Now fix q at a positive value and increase v starting at $v = 0$. At very low values of v, the region specializes in the production of the services-intensive good 1. With increasing values of v, it reaches a point of diversification at a positive v and eventually reaches the point of specialization in the production of good 2 at an even greater value of v. Since this holds for every fixed positive value of q (as well as the above analysis of rising values of q),[1] this implies that the lines $S(2)$ and $S(1)$ start at the origin. Accordingly, the pattern of specialization in (q, v)-space looks qualitatively as shown in Fig. C1.

Whether $S(2)$ is always steeper than $S(1)$ depends on a mixture of up to second order derivatives of both production functions. No assumptions about the relative size of the involved magnitudes have been made, because these would not have any clear economic meaning. For simplicity, it is assumed that $S(2)$ is actually steeper than $S(1)$.[2] Nothing fundamental is changed without this assumption.

2 The Isocline $\dot{q} = 0$

For given values of v and c, the function $\dot{q} = q^n Z^1(pq^n, v, c) - \rho q$ [eq. (5)] is shown in Fig. C2. The region between the kinks corresponds to diversified production. Since the second order derivative of $\dot{q}$ with respect to q depends, among others, on a mixture of up to third order derivatives of the production function, no reasonable assumption generating a concave curve is possible. Hence, multiple equilibria will exist in the general case. Apart from this possibility, however, if it is assumed that there is at least one positive root

[1] No proof of this result is given, since it is intuitively clear from considering the well-known Harrod-Johnson diagram. Under the assumptions made about the production functions, the functions $k_i(\omega)$ are monotonously increasing and start at the origin. In this diagram, given any pq^n, it is then always possible to choose values of k that yield specialization in the production of one of both goods or diversification.

[2] The comparison of the slopes of both curves reveals that $-npX_p^1/(qX_v^1) > -npX_p^2/(qX_v^2)$ if both derivatives are calculated at the same point. To prove this, note that the above inequality is equivalent to $X_v^2/X_v^1 < X_p^2/X_p^1 = -p$ in this case, which may be rewritten as $R_v = pX_v^1 + X_v^2 > 0$. The slopes have to be calculated at different values of v, however; while X_v^i is independent of v in the region of diversification, X_p^i changes with v depending on the mentioned mixture of up to second order derivatives of both production functions.

of $\dot{q} = 0$ with $\partial \dot{q} / \partial q \neq 0$, the qualitative features are determined as in Fig. C2 (notice that $\dot{q} = -\rho q$ if $x_1 = 0$). Regarding the numerical examples leading to Figures 3 and 4, however, the curve is concave under diversification. Depending on the respective values of v, c and p, $\dot{q}$ will shift and may lie completely below the q-axis, e.g.

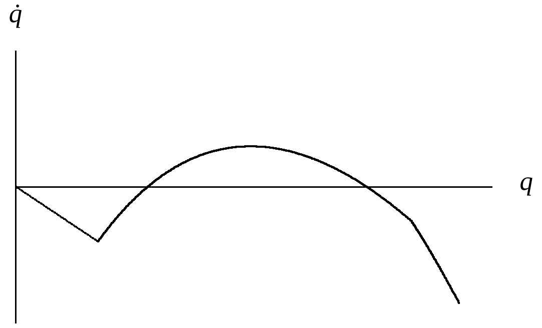

Fig. C2. $\dot{q}$ as a function of q

If the curve is placed as in Fig. C2, one starts at low values of q producing only good 2, reaches diversification at a higher value of q with $\dot{q}$ still negative, and crosses two roots of $\dot{q} = 0$ before reaching specialization in the production of good 1. For parameterically varying values of v, the graph in Fig. C2 will shift accordingly. If v rises, diversification starts at a higher value of q and the part of the curve between the kinks shifts downwards (cf. Fig. C1 and the Rybczynski theorem). Specialization in the production of good 1 occurs at a higher value of q. In fact, the region of diversification between the kinks expands, since it has been assumed that $S(2)$ is always steeper than $S(1)$ in Fig. C1. The opposite effects occur if v falls.

In Fig. C4, which corresponds to Fig. 2 neglecting the vector field, this information is useful in determining the shape of the isocline $\dot{q} = 0$ in (q, v)-space. In the region of specialization in the production of good 2, $\dot{q}$ must be negative for all $q > 0$. Crossing the curve $S(2)$ eastwards, the production of good 1 is taken up. If v is sufficiently low, the region's production of good 1 will be high enough to reach a point where $\dot{q} = 0$. The maximum of the curve $\dot{q} = 0$ in Fig. C4 corresponds to the case where $\dot{q}$ in Fig. C2 has just one tangential point with the q-axis. For lower values of v, the production of good 1 rises and the case shown in Fig. C2 arises. Then, in Fig. C4, $\dot{q}$ becomes positive when going further eastwards until the second root of $\dot{q} = 0$ is reached. For low enough values of v, this second root may lie below the curve $S(1)$, implying that the region specializes in the production of good 1 here. The shape of $\dot{q} = 0$ below $S(1)$ depends on the fact that, with specialization in the production of good 1, the amount produced is now *rising* with v instead of *falling* as implied by Rybczynski's theorem in the case of diversification. That the locus $\dot{q} = 0$ must begin and end at the origin may be seen as follows: the first intersection with the q-axis in Fig. C2 occurs always in the region of diversification, while for low values of v the second occurs in the region of specialization in the production of good 1. The latter intersection point is $(z_1^{\max} / \rho)^{1/(1-n)}$, and it approaches zero if $z_1^{\max} \to 0$ as $v \to 0$. For lower values of q, an intersection occurs in the region of diversification, that is, $\dot{q} = 0$ between the lines $S(1)$ and $S(2)$ in Fig. C4.

3 The Isocline $\dot{v} = 0$

Now turn to the differential equation (6), $\dot{v} = \alpha[R(pq^n, v, c) - pc - y^* v]$, and fix c and q. The lower part of Fig. C3 shows the difference of $\alpha R(pq^n, v, c)$ and the sum of $\alpha y^* v$ and the constant αpc (shown in the upper part of the figure). For given values of c and q, the Inada-conditions imply that there are either two roots or no roots (especially, if q is too low and thus $R(pq^n, v, c)$ is too small) of $\dot{v} = 0$, unless the curve $\dot{v}$ is tangential to the v-axis.

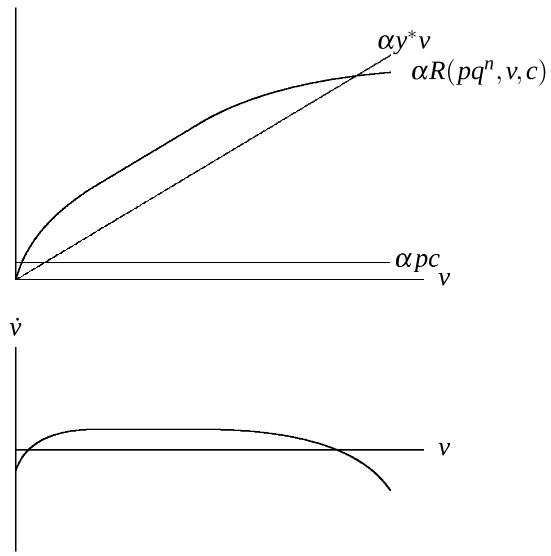

Fig. C3. $\dot{v}$ as a function of v

In case of Fig. C3, one starts at low values of v producing only good 1, reaches diversification at a higher value of v with $\dot{v}$ positive, and specialization in the production of good 2 with $\dot{v}$ still positive, before $\dot{v}$ eventually becomes negative. If q changes paremeterically, the graph in Fig. C3 will shift accordingly. The resulting changes of the respective location and slope in the three different regions are calculated by the following derivatives:

$$\frac{\partial \dot{v}}{\partial q} = \alpha R_q \begin{cases} > 0 & : \text{specialization good 1} \\ > 0 & : \text{diversification} \\ = 0 & : \text{specialization good 2} \end{cases}$$

$$\frac{\partial^2 \dot{v}}{\partial v \partial q} = \alpha R_{vq} \begin{cases} > 0 & : \text{specialization good 1} \\ < 0 & : \text{diversification} \\ = 0 & : \text{specialization good 2} \end{cases}$$

Note that q has no effect on R and R_v if the region specializes in the production of good 2. The sign of αR_{vq} in the case of diversification follows from $R_{vq} = R_{qv} = R_{pv} np/q = npX_v^1/q < 0$.

With respect to the lower part of Fig. C3, a higher value of q will move the first two parts of the graph upwards; the slope of the first part rises while the slope of the second part falls. Diversification starts at a higher value of v. Even specialization in the production of good 2 occurs at a higher value of v and the region of diversification expands (cf. Fig. C1). The last part of the graph with specialization in the production of good 2 is not affected by q, but it is getting smaller since it occurs at a higher value of v than before. The opposite effects occur if q falls.

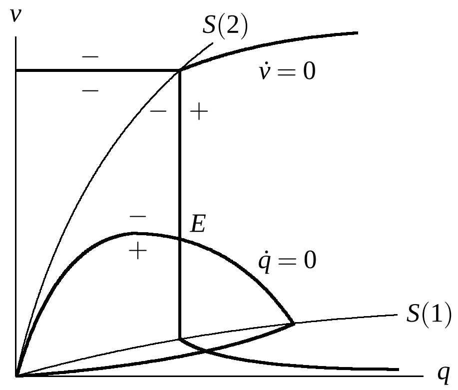

Fig. C4. The isoclines $\dot{q} = 0$, $\dot{v} = 0$, and the pattern of specialization

In Fig. C4, this information is used to determine the shape of the isocline $\dot{v} = 0$ in (q, v)-space for a special case in which the level of government spending at the diversified equilibrium E is statically efficient. The appearance of the isocline varies considerably depending on the respective position of the function $\dot{v}$ in Fig. C3 for varying values of q. In case of Fig. C4, for low values of q, the function $\dot{v}$ in Fig. C3 lies below the v-axis and has a tangential point with the v-axis in the region of specialization in the production of good 2. With rising values of q, the linear part of $\dot{v}$ in Fig. C3 is getting flatter and moves upwards; the region of specialization in the production of good 2 becomes smaller. The tangential point, however, does not disappear until this linear part coincides with the v-axis and has the tangential point as its right boundary. This explains the horizontal and the vertical linear parts of the locus $\dot{v} = 0$ in Fig. C4. As q rises again, the linear part in Fig. C3 shifts more upwards and expands while its slope is getting negative and there are two intersections of $\dot{v}$ with the v-axis, one with specialization in the production of good 1, and one with diversification. These intersection points move to the left respectively to the right as q rises again. This explains the two nonlinear parts of $\dot{v} = 0$ in Fig. C4.[3]

References

Herberg, H. (1969): On the Shape of the Transformation Curve in the Case of Homogenous Production Functions, *Zeitschrift für die gesamte Staatswissenschaft*, 125, 202–210.

[3]The isocline $\dot{v} = 0$ has a negative slope below $S(1)$, while the slope of the nonlinear part in the region of diversification is positive. E.g., the part below $S(1)$ can be shown to be downward sloping if $R_c - p = pq^n \partial f_1/\partial c - p < 0$ at $\dot{v} = 0$, which applies if $pq^n \partial f_1/\partial v > y^*$. This is the case if $\dot{v}$ in Fig. C3 intersects the v-axis from below. Thus, part of $\dot{v} = 0$ is upward sloping below $S(1)$ only if $\dot{v}$ in Fig. C3 cuts the v-axis in the region of specialization in the production of good 1 both from below and from above.